Indian Statistical Institute

Mid-Semestral Examination 2016-2017

B.Math First Year

Analysis II

Time : 3 Hours   Date : 20.02.2017   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) $\mathcal{R}[a,b] :=$ Riemann integrable functions on $[a,b]$.

Q1. (10 marks) Give an example of a bounded function on $[a,b]$, $a<b$, which is not Riemann integrable.

Q2. (15 marks) Let $f:[a,b]\to\mathbb{R}$ be a bounded function and

$$\underline{\int_a^b} f > 0.$$

Prove that there exists an interval $I \subseteq [a,b]$ such that $f>0$ on I.

Q3. (15 marks) Let $f \in \mathcal{R}[a,b]$. Prove that $|f| \in \mathcal{R}[a,b]$ and

$$|\int_a^b f| \leq \int_a^b |f|.$$

Q4. (10 marks) Let $f \in \mathcal{R}[a,b]$. Prove that for

$$F(x) = \int_a^x f \qquad (x \in [a,b]),$$

there is $M>0$ such that

$$|F(x)-F(y)| \leq M|x-y| \qquad (x,y \in [a,b]).$$

Q5. (10 marks) Suppose that (X,d) is a metric space and $\{S_\alpha\}_{\alpha\in\Lambda}$ is a collection of subsets of X. Prove that

$$\cup_\alpha \overline{S_\alpha} \subseteq \overline{\cup_\alpha S_\alpha} = \overline{\cup_\alpha \overline{S_\alpha}}.$$

Q6. (15 marks) Prove or disprove the following:

(i) A discrete metric space is complete.

(ii) An infinite subset of a metric space has a limit point.

(iii) A non-empty complete metric space without isolated points is uncountable.

Q7. (15 marks) Prove that $B[0,1]$ (with uniform metric) is not separable.

Q8. (10 marks) Prove that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X,d), then $\{d(x_n,y_n)\}$ is a convergent sequence.

Q9. (15 marks) Prove that a closed interval cannot be expressed as the union of a countable family of disjoint nonempty closed sets.